R-MATRICES AND HOPF ALGEBRA QUOTIENTS

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ABSTRACT. We study a natural construction of Hopf algebra quotients canonically associated to an R-matrix in a finite dimensional Hopf algebra. We apply this construction to show that a quasitriangular Hopf algebra whose dimension is odd and square-free is necessarily cocommutative.

1. INTRODUCTION AND MAIN RESULTS

Quasitriangular Hopf algebras have been introduced by Drinfeld in [7]; these are Hopf algebras H endowed with a universal solution of the Quantum Yang-Baxter Equation $R \in H \otimes H$, called an R-matrix. Such an R-matrix gives rise to a map $\Phi_R : H^* \to H$,

$$\Phi_R(f) := f(R^{(2)}r^{(1)})R^{(1)}r^{(2)}, \quad (R = r),$$

also studied by Drinfeld in his paper [8], and later interpreted in categorical terms by S. Majid. Indeed, the *transmutation* map Φ_R turns out to be a morphism of *braided* Hopf algebras, after Majid's work.

Two extreme classes of quasitriangular Hopf algebras appear in relation with the map Φ_R: *triangular* Hopf algebras are those for which Φ_R is trivial, while *factorizable* Hopf algebras are those for which Φ_R is an isomorphism.

Throughout this paper we shall work over an algebraically closed field k of characteristic zero.

Finite dimensional triangular Hopf algebras have been recently classified by Etingof and Gelaki, based on results of Deligne on symmetric categories [12]. On the other hand, several important results are known to be true for factorizable Hopf algebras [23, 9, 25]. In particular, factorizable Hopf algebras are useful in the study of invariants of 3-manifolds. The Drinfeld double of a finite dimensional Hopf algebra is always factorizable.

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In this paper we give a construction of certain canonical Hopf algebra quotients of a quasitriangular Hopf algebra. Our construction is based on properties of transmutation due to Majid [14, Chapter 9.4]. Explicitly, the construction goes as follows: any Hopf subalgebra of the dual is 'transmuted' into a normal coideal subalgebra, and this last one is a sort of 'kernel' of a unique Hopf algebra quotient, in view of the correspondence due to Takeuchi [27]. This result is contained in Theorem 4.2. Several applications are also given in Section 4.

An important special case of Theorem 4.2 is stated in Theorem 4.4. It says that, in a sense, every quasitriangular Hopf algebra is an 'extension' of the image of the map Φ_R by a canonical triangular Hopf algebra.

We also discuss conditions, involving the R-matrix, that guarantee normality of a quasitriangular Hopf algebra quotient. In this context, we give a necessary and sufficient condition for a map $H \to B$ to be normal; see Proposition 3.10. Combining these results with the classification in [2, 10, 12] we prove in Theorem 4.11 that if (H, R) is a quasitriangular Hopf algebra with $R \in kG(H) \otimes kG(H)$, then H must be an extension of a dual group algebra by a twisting of a modified supergroup algebra.

We apply the results in Section 4 to prove a classification result for quasitriangular Hopf algebras. The following open question appeared in [1, Question 6.5].

Question 1.1. *Let $N \geq 1$ be a square-free integer. Are all Hopf algebras of dimension N over k necessarily semisimple?*

In the paper [19] we proved that a quasitriangular Hopf algebra of dimension pq, where p and q are odd prime numbers is semisimple and therefore cocommutative. The following theorem generalizes this result, and gives a partial answer to Question 1.1 in the quasitriangular case.

Theorem 1.2. *Let (H, R) be a quasitriangular Hopf algebra. Assume that $\dim H$ is odd and square-free. Then H is semisimple.*

Moreover, $H \simeq kG$ where the group G is a semidirect product $G = \widehat{\Gamma} \rtimes F$, with Γ cyclic. The R-matrix corresponds under this isomorphism to an F-invariant non-degenerate bicharacter $\langle\, ,\, \rangle : \Gamma \times \Gamma \to k^\times$ such that $\langle\, ,\, \rangle \langle\, ,\, \rangle^{\mathrm{op}}$ is also non-degenerate.

Theorem 1.2 will be proved in Section 5.

The paper is organized as follows. In Section 2 we recall the definition and basic properties of quasitriangular Hopf algebras. In Section

3 we discuss the question of normality of Hopf algebra quotients in the quasitriangular case. Section 4 contains our main results on the existence of Hopf algebra quotients attached to an R-matrix, and some of its consequences. Finally, in Section 5 we apply the results in previous sections to the classification of quasitriangular Hopf algebras of square-free odd dimension.

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1.1. **Preliminaries and notation.** Our references for the theory of Hopf algebras are [17, 24]. For a Hopf algebra H, the antipode of H will be denoted by $\mathcal{S}$. The group of group-like elements in H will be denoted $G(H)$.

For an element $Z = \sum_i x_i \otimes y_i \in H \otimes H$, the induced linear map $H^* \to H$, $p \to \sum_i \langle p, x_i \rangle y_i$ will be denoted by f_Z. Thus, if $\dim H$ is finite, $Z \mapsto f_Z$ gives an isomorphism $H \otimes H \to \mathrm{Hom}(H^*, H)$.

Also for such Z, the notation Z_{21} will be used to indicate the element $\tau(Z)$, where $\tau : H \otimes H \to H \otimes H$ is the usual flip map $\tau(x \otimes y) = y \otimes x$.

Recall that Hopf algebra H is called *simple* if it contains no proper normal Hopf subalgebras in the sense of [17, 3.4.1]; H is called *semisimple* (respectively, *cosemisimple*) if it is semisimple as an algebra (respectively, if it is cosemisimple as a coalgebra).

2. Quasitriangular and factorizable Hopf algebras

Let (H, R) be a quasitriangular Hopf algebra over k. That is, $R \in H \otimes H$ is an invertible element, called an *R-matrix*, fulfilling the following conditions:

(QT1) $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$.
(QT2) $(\epsilon \otimes \mathrm{id})(R) = 1$.
(QT3) $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$.
(QT4) $(\mathrm{id} \otimes \epsilon)(R) = 1$.
(QT5) $\Delta^{\mathrm{cop}}(h) = R\Delta(h)R^{-1}$, $\forall h \in H$.

The following relations with the antipode of H are well-known:

$$(\mathcal{S} \otimes \mathrm{id})(R) = R^{-1} = (\mathrm{id} \otimes \mathcal{S}^{-1})(R), \quad (\mathcal{S} \otimes \mathcal{S})(R) = R. \tag{2.1}$$

There are Hopf algebra maps $f_R : H^{*\mathrm{cop}} \to H$ and $f_{R_{21}} : H^* \to H^{\mathrm{op}}$ given, respectively, by

$$f_R(p) = \langle p, R^{(1)} \rangle R^{(2)}, \quad f_{R_{21}}(p) = \langle p, R^{(2)} \rangle R^{(1)},$$

for all $p \in H^*$. Here, and elsewhere, we use the shorthand notation $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$. The images of these maps will be indicated by H_+ and H_-, respectively.

Thus H_+, H_- are finite dimensional Hopf subalgebras of H and $H_+ \simeq (H_-^*)^{\mathrm{cop}}$. The *rank* of R is defined as $\operatorname{rk} R := \dim H_+ = \dim H_-$.

We shall denote by $H_R = H_-H_+ = H_+H_-$ the minimal quasitriangular Hopf subalgebra of H. See [20].

Example 2.1. (See [5]). Let $H = kG$ be a cocommutative Hopf algebra, where $G = G(H)$ is a finite group. Suppose (H, R) is a quasitriangular structure on H. Since $H^{*\mathrm{cop}}$ and H^* are commutative, then H_+ and $H_- \simeq {H_+}^{*\,\mathrm{cop}}$ are commutative and cocommutative Hopf subalgebras of H. In particular, the Drinfeld double $D(H_+)$ is a commutative Hopf algebra.

Write $H_R = k\Gamma$, where $\Gamma \subseteq G$ is a subgroup. In view of [20], there is a Hopf algebra projection $D(H_+) \to H_R = k\Gamma$, implying that Γ is an *abelian* subgroup.

The set $\{e_a\}_{a\in\widehat{\Gamma}}$ is a complete set of primitive orthogonal idempotents in $k\Gamma$, where $e_a = \dfrac{1}{|\Gamma|}\sum_{g\in\Gamma} a(g^{-1})g$, $a \in \widehat{\Gamma}$. Then R-matrix $R \in k\Gamma \otimes k\Gamma$ can be written as

$$R = \sum_{a,b\in\widehat{\Gamma}} \rho(a,b) e_a \otimes e_b, \tag{2.2}$$

for a unique bilinear form $\rho : \widehat{\Gamma} \times \widehat{\Gamma} \to k^\times$. Now Condition (QT5) implies that Γ is a *normal* subgroup of G, and moreover $\rho : \widehat{\Gamma}\times\widehat{\Gamma} \to k^\times$ is an $\operatorname{ad} G$-invariant bilinear form on $\widehat{\Gamma}$.

Conversely, it is not difficult to see that every pair (Γ, ρ), with $\Gamma \subseteq G$ a normal abelian subgroup and $\rho : \widehat{\Gamma} \times \widehat{\Gamma} \to k^\times$ an $\operatorname{ad} G$-invariant bilinear form, determines a unique quasitriangular structure on $H = kG$ via the formula (2.2).

Recall from [23] that the quasitriangular Hopf algebra (H, R) is called *factorizable* if the map $\Phi_R : H^* \to H$ is an isomorphism, where

$$\Phi_R(p) = \langle p, Q^{(1)}\rangle Q^{(2)}, \quad p \in H^*; \tag{2.3}$$

here, $Q = Q^{(1)} \otimes Q^{(2)} = R_{21}R \in H \otimes H$. In other words, $\Phi_R = f_Q$.

Remark 2.2. The image $\Phi_R(H^*)$ coincides with the subspace of H spanned by the right tensorands of $Q \in H \otimes H$.

Note that $\Phi_R(H^*) \subseteq H_R$. So every factorizable Hopf algebra is minimal quasitriangular in the sense of [20].

If on the other hand $\Phi_R = \epsilon 1$ (or equivalently, $R_{21}R = 1 \otimes 1$), then (H, R) is called *triangular*. Finite dimensional triangular Hopf algebras were completely classified in [12].

We may also consider the map ${}_R\Phi = f_{Q_{21}} : H^* \to H$, given by

$$
{}_R\Phi(p) = \langle p, Q^{(2)} \rangle Q^{(1)}, \quad p \in H^*. \tag{2.4}
$$

Lemma 2.3. ${}_R\Phi = \mathcal{S}\Phi_R\mathcal{S}$. *In particular,* ${}_R\Phi(H^*) = \mathcal{S}\Phi_R(H^*)$.

Proof. We have $(\mathcal{S}\otimes\mathcal{S})(R) = R$, and therefore also $(\mathcal{S}\otimes\mathcal{S})(R_{21}) = R_{21}$. Denote as before $Q = R_{21}R$, then

$$
(\mathcal{S}\otimes\mathcal{S})(Q) = (\mathcal{S}\otimes\mathcal{S})(R)(\mathcal{S}\otimes\mathcal{S})(R_{21}) = RR_{21} = Q_{21}.
$$

Therefore, for all $p \in H^*$,

$$
{}_R\Phi(p) = \langle p, \mathcal{S}(Q^{(1)})\rangle \mathcal{S}(Q^{(2)}) = \mathcal{S}(\langle \mathcal{S}(p), Q^{(1)}\rangle Q^{(2)}) = \mathcal{S}(\Phi_R(\mathcal{S}(p))).
$$

□

Remark 2.4. Suppose that H is semisimple. Let $R(H) \subseteq H^*$ and $Z(H) \subseteq H$ denote, respectively, the character algebra and the center of H; so that $R(H)$ coincides with the subalgebra of cocommutative elements in H^*. Then $\Phi_R : R(H) \to Z(H)$ is an algebra map [8].

Note that for all $p, q \in R(H)$, $\langle p \otimes q, R_{21}R\rangle = \langle q \otimes p, R_{21}R\rangle$; and therefore, $\langle p, \Phi_R(q)\rangle = \langle q, \Phi_R(p)\rangle$.

So that the bilinear form $[\ ,\ ] : R(H) \otimes R(H) \to k$, $[p, q] = \langle p \otimes q, R_{21}R\rangle$ is symmetric. Moreover, (H, R) is factorizable if and only if $[\ ,\ ]$ is non-degenerate.

Let $\widehat{H} = \{\chi_0 = \epsilon, \ldots, \chi_n\}$ be the set of irreducible characters of H. Let

$$
s_{ij} := \langle \chi_i, \Phi_R(\chi_j)\rangle = \langle \chi_j, \Phi_R(\chi_i)\rangle, \quad 0 \leq i, j \leq n.
$$

Then (H, R) is factorizable if and only if the matrix $S = (s_{ij})_{0\leq i,j\leq n}$ is non-degenerate, if and only if the category H-mod of finite dimensional H-modules is *modular.*

Example 2.5. Recall that for a finite dimensional Hopf algebra A, its Drinfeld double is a quasitriangular Hopf algebra. $D(A) = A^{*\,\mathrm{cop}} \otimes A$ as a coalgebra, with a canonical R-matrix $\mathcal{R} = \sum_i a^i \otimes a_i$, where $(a_i)_i$ is a basis of A and $(a^i)_i$ is the dual basis.

It is well-known that the Drinfeld double $(D(A), \mathcal{R})$ is factorizable [23]. We have $D(A)_+ = A$, $D(A)_- = A^{*\mathrm{cop}}$.

Our next result relates factorizability with semisimplicity of a quasi-triangular Hopf algebra. We showed in [19, Corollary 2.5] that if (H, R)

is a factorizable odd-dimensional Hopf algebra such that all proper Hopf subalgebras of H are semisimple, then H is itself semisimple.

The following lemma gives a further criterion in this direction. It will be used in Section 5.

Lemma 2.6. *Let (H, R) be a factorizable Hopf algebra. Assume in addition that $\dim H$ is odd and $\operatorname{rk} R = \dim H$. Then H is semisimple.*

Proof. Recall that H is semisimple if and only if $\operatorname{Tr} \mathcal{S}^2 \neq 0$ [21]. Therefore if $\mathcal{S}^4 = \mathrm{id}$ and $\dim H$ is odd, then H must be semisimple.

Let $g \in G(H)$, $\alpha \in G(H^*)$ be the modular elements of H. By [22], we have $\alpha = 1$. On the other hand, the assumption on the rank of R says that f_R is an isomorphism. Therefore $f_{R_{21}}(\alpha) = g^{-1}$ by [13, Corollary 2.10]. This implies that also $g = 1$. Then $\mathcal{S}^4 = \mathrm{id}$ [17, Proposition 10.1.14]. Since $\dim H$ is odd, this implies that H is semisimple as claimed. □

3. Normal quotients of quasitriangular Hopf algebras

Let H and B be finite dimensional Hopf algebras over k and let $\pi : H \to B$ be a surjective Hopf algebra map.

Then $H^{\mathrm{co}\,\pi} := \{h \in H : (\mathrm{id} \otimes \pi)\Delta(h) = h \otimes 1\}$, is a left coideal subalgebra of H. Similarly, ${}^{\mathrm{co}\,\pi}H := \{h \in H : (\pi \otimes \mathrm{id})\Delta(h) = 1 \otimes h\}$ is a right coideal subalgebra of H. We shall use the notation $H^{\mathrm{co}\,B} := H^{\mathrm{co}\,\pi}$, ${}^{\mathrm{co}\,B}H := {}^{\mathrm{co}\,\pi}H$, when no ambiguity arises.

Moreover, $H^{\mathrm{co}\,\pi}$ is stable under the left adjoint action of H, defined as

$$\mathrm{ad}_h(a) := h_1 a \mathcal{S}(h_2),$$

for all $h, a \in H$. A subspace of H stable under the adjoint action will be called *normal*.

Remark 3.1. Consider the left (respectively, right) action of B^* on H given by $f \rightharpoonup h = \langle f, \pi h_2 \rangle h_1$, (respectively, $h \leftharpoonup f = \langle f, \pi h_1 \rangle h_2$). Then we have $H^{\mathrm{co}\,\pi} = {}^{B^*}H$ and ${}^{\mathrm{co}\,\pi}H = H^{B^*}$.

Definition 3.2. We shall say that a sequence $K \subseteq H \xrightarrow{\pi} B$ is *exact* if π is surjective Hopf algebra map and $K = H^{\mathrm{co}\,B}$ or $K = {}^{\mathrm{co}\,B}H$.

The surjection π is called *normal* if $H^{\mathrm{co}\,\pi} = {}^{\mathrm{co}\,\pi}H$. In this case $K = H^{\mathrm{co}\,\pi}$ is a (normal) Hopf subalgebra of H and there is an exact sequence of Hopf algebras $1 \to K \to H \to B \to 1$. Furthermore, $B \simeq H/HK^+$ as Hopf algebras.

Lemma 3.3. *The following are equivalent:*

(i) *The surjection π is normal.*

(ii) $H^{\mathrm{co}\,\pi}$ *is a right coideal of* H.
(iii) $H^{\mathrm{co}\,\pi}$ *is a subcoalgebra of* H.

Proof. We omit the details of the proof. □

Remark 3.4. Suppose that A is a Hopf subalgebra of H. Then $A \subseteq H^{\mathrm{co}\,\pi}$ if and only if $A \subseteq {}^{\mathrm{co}\,\pi}H$ if and only if $\pi|_A = \epsilon$.

From now on (H, R) will be a finite dimensional quasitriangular Hopf algebra and $q : H \to B$ will be a surjective Hopf algebra map.

We aim to give a necessary and sufficient condition for the map q to be normal. We point out that a result in this direction has been obtained by Masuoka in [16, Corollary 5], where a sufficient condition for normality of a special kind of quotient is given. Contrary to our approach, this condition is independent of the R-matrix involved.

Recall that $H_+ \subseteq H$ denotes the Hopf subalgebra of H spanned by the right tensorands of R.

Proposition 3.5. *Suppose $R \in H \otimes H^{\mathrm{co}\,B}$. Then $q : H \to B$ is normal.*

Proof. We need to show that $H^{\mathrm{co}\,B}$ is a subcoalgebra, hence a Hopf subalgebra, of H.

We know that $\Delta(H^{\mathrm{co}\,B}) \subseteq H \otimes H^{\mathrm{co}\,B}$. The assumption $R \in H \otimes H^{\mathrm{co}\,B}$ implies that also $R^{-1} = (\mathcal{S} \otimes \mathrm{id})(R) \in H \otimes H^{\mathrm{co}\,B}$. Let $h \in H^{\mathrm{co}\,B}$, so that $\Delta(h) \in H \otimes H^{\mathrm{co}\,B}$. On the other hand, we have

$$\Delta^{\mathrm{cop}}(h) = R\Delta(h)R^{-1} \in H \otimes H^{\mathrm{co}\,B}.$$

This implies that $\Delta(h) \in (H \otimes H^{\mathrm{co}\,B}) \cap (H^{\mathrm{co}\,B} \otimes H) = H^{\mathrm{co}\,B} \otimes H^{\mathrm{co}\,B}$. Therefore $H^{\mathrm{co}\,B}$ is a subcoalgebra of H as claimed. □

Corollary 3.6. *Suppose that $H_+ \subseteq H^{\mathrm{co}\,B}$. Then $q : H \to B$ is normal.*

Proof. H_+ is the smallest Hopf subalgebra of H with $R \in H \otimes H_+$. The corollary follows from Proposition 3.5. □

Corollary 3.7. *Suppose* $\dim B$ *is relatively prime to the rank of R. Then $q : H \to B$ is normal.*

Proof. We have $\mathrm{rk}\, R = \dim H_+$. Thus the assumption implies that $q|_{H_+} = \epsilon$ and hence $H_+ \subseteq H^{\mathrm{co}\,B}$. Therefore q is normal in view of Corollary 3.6. □

The following is an application of Corollary 3.6 to the Drinfeld double. See Example 2.5.

Corollary 3.8. *Let A be a finite dimensional Hopf algebra and let $q : D(A) \to B$ be a surjective Hopf algebra map. Suppose that $q|_A = \epsilon$. Then q is normal.* □

Note that (B, R_q) is quasitriangular with R-matrix $R_q = (q \otimes q)(R)$. Note in addition that $f_{R_q} = q f_R q^* : B^* \to B$.

Identify B^* with a Hopf subalgebra of H^* by means of the transpose map $q^* : B^* \hookrightarrow H^*$. Recall that $H^{\operatorname{co} B}$ coincides with the space of B^*-invariants of H under the left regular action $\rightharpoonup: B^* \otimes H \to H$, while ${}^{\operatorname{co} B}H$ coincides with the B^*-invariants under the right regular action $\leftharpoonup: H \otimes B^* \to H$.

Lemma 3.9. *Let $p \in B^*$, $a \in H^{\operatorname{co} B}$. Then $a \leftharpoonup p = \operatorname{ad}(f_{R_{21}} p)(a)$.*

Proof. By (QT5) we have $a_2 \otimes a_1 = R^{(1)} a_1 \mathcal{S}(r^{(1)}) \otimes R^{(2)} a_2 r^{(2)}$, since $R^{-1} = (\mathcal{S} \otimes \operatorname{id})(R)$. Therefore,

$$\begin{aligned} a \leftharpoonup p = \langle p, a_1 \rangle a_2 &= \langle p, R^{(2)} a_2 r^{(2)} \rangle R^{(1)} a_1 \mathcal{S}(r^{(1)}) \\ &= \langle p_1, R^{(2)} \rangle \langle p_2, a_2 \rangle \langle p_3, r^{(2)} \rangle R^{(1)} a_1 \mathcal{S}(r^{(1)}) \\ &= f_{R_{21}}(p_1)(p_2 \rightharpoonup a) \mathcal{S}(f_{R_{21}} p_3) \\ &= f_{R_{21}}(p_1) a \mathcal{S}(f_{R_{21}} p_2) = f_{R_{21}}(p)_1 a \mathcal{S}((f_{R_{21}} p)_2) \\ &= \operatorname{ad}(f_{R_{21}} p)(a), \end{aligned}$$

where we used that $f_{R_{21}}$ is a coalgebra map. This proves the lemma. $\square$

Proposition 3.10. *The following are equivalent:*

(i) *The map $q : H \to B$ is normal.*
(ii) $f_{R_{21}}(B^*) \subseteq (H^{\operatorname{co} B})'$.

Here $(H^{\operatorname{co} B})'$ denotes the centralizer of $H^{\operatorname{co} B}$ in H.

Proof. (i) $\Rightarrow$ (ii). By Remark 3.1 and Lemma 3.9, the assumption implies that $\operatorname{ad}(f_{R_{21}} p)(a) = \epsilon(p) a$, for all $p \in B^*$, $a \in H^{\operatorname{co} B}$. Then $f_{R_{21}}(B^*) \subseteq (H^{\operatorname{co} B})'$.

(ii) $\Rightarrow$ (i). Let $p \in B^*$, $a \in H^{\operatorname{co} B}$. By Lemma 3.9, we have

$$a \leftharpoonup p = \operatorname{ad}(f_{R_{21}} p)(a) = \epsilon(p) a.$$

This shows that $H^{\operatorname{co} B}$ is invariant under the right regular action of B^*. In view of Remark 3.1 this implies that ${}^{\operatorname{co} B}H = H^{\operatorname{co} B}$, and the normality of q. $\square$

Remark 3.11. Note that condition (ii) in Proposition 3.10 holds in any of the following cases:

(i) $f_R|_{B^*} = \epsilon$, or
(ii) $H^{\operatorname{co} B} \subseteq Z(H)$.

Lemma 3.12. *Let $A \subseteq H^*$ be a Hopf subalgebra. Suppose that $A \subseteq (H^{*\operatorname{cop}})^{\operatorname{co} f_R}$. Then A is normal in H^*.*

In particular, if $(\dim A, \operatorname{rk} R) = 1$, then A is normal in H^*. Thus we recover the statement in Corollary 3.7.

Proof. Consider the quotient $H \to B = A^*$. The assumption $A \subseteq (H^{*\mathrm{cop}})^{\mathrm{co}\, f_R}$ implies that $f_R|_{B^*} = \epsilon$. The lemma follows from Proposition 3.10, in view of Remark 3.11 (i). □

Corollary 3.13. *Assume H is simple. Then the restriction $f_R|_{G(H^*)}$ is injective. In particular, $|G(H^*)|$ divides* $\operatorname{rk} R$.

Proof. Note that the intersection $kG(H^*) \cap (H^{*\mathrm{cop}})^{\mathrm{co}\, f_R}$ coincides with the group algebra of the kernel of the restriction $f_R|_{G(H^*)}$. By Lemma 3.12, if H is simple, the restriction $f_R|_{G(H^*)}$ must be injective, as claimed. Moreover, in this case $f_R|_{G(H^*)} \simeq kG(H^*)$ is a Hopf subalgebra of H_+ implying that $|G(H^*)|$ divides $\operatorname{rk} R$. □

Proposition 3.14. *The following are equivalent:*

(i) $R_q = 1 \otimes 1$,
(ii) $f_R(B^*) \subseteq H^{\mathrm{co}\, q} \cap {}^{\mathrm{co}\, q}H$,
(iii) $f_{R_{21}}(B^*) \subseteq H^{\mathrm{co}\, q} \cap {}^{\mathrm{co}\, q}H$.

Proof. Clear. See Remark 3.4. □

4. Hopf algebra quotients arising from R-matrices

Let H be a finite dimensional Hopf algebra. Then H is faithfully flat over its left coideal subalgebras [26]. By [27, Theorem 3.2], the maps

$$I \mapsto H^{\mathrm{co}\, H/I}, \qquad L \mapsto HL^+,$$

give rise to inverse bijective correspondences between:

(a) the set of Hopf ideals I of H, and
(b) the set of normal left coideal subalgebras L of H.

From now on, let (H, R) be a finite dimensional quasitriangular Hopf algebra. In this section we shall combine the above with the transmutation theory of Majid [14, Chapter 9.4].

By [14, Example 9.4.9] there is a braided Hopf algebra $\underline{H}$ in the braided category ${}_H\mathcal{M}$ of left H-modules, where $\underline{H} = H$ as algebras, with comultiplication

$$\underline{\Delta}(a) = a_1 \mathcal{S}(R^{(2)}) \otimes \mathrm{ad}_{R^{(1)}}(a_2), \tag{4.1}$$

and H acts on $\underline{H}$ via the left adjoint action:

$$\mathrm{ad}_h(a) = h_1 a \mathcal{S}(h_2).$$

Dually [14, Example 9.4.10], there is a braided Hopf algebra structure $\underline{H^*}$ in $\mathcal{M}^{H^*} = {}_H\mathcal{M}$ where $\underline{H^*} = H^*$ as coalgebras, with multiplication

$$p\underline{.}q = (\mathcal{S}(p_1)p_3 \otimes \mathcal{S}(q_1))\,(R)\,p_2q_2, \tag{4.2}$$

and H^* coacts on $\underline{H^*}$ via the right adjoint coaction: $\rho : \underline{H^*} \to \underline{H^*} \otimes H^*$,

$$\rho(p) = p_2 \otimes \mathcal{S}(p_1)p_3. \tag{4.3}$$

In this context, the map Φ_R given by (2.3) becomes a morphism of braided Hopf algebras $\Phi_R : \underline{H^*} \to \underline{H}$. In other words, Φ_R is a left H-module map that transforms the product (4.2) into the product of H and the coproduct of H^* into the coproduct (4.1). See [14, Propositions 2.1.14 and 7.4.3].

Lemma 4.1. *Let $C \subseteq H^*$ be a subcoalgebra. Then $\Phi_R(C) \subseteq H$ is a normal left coideal of H.*

Proof. Note that a subcoalgebra $C \subseteq H^*$ is an H-subcomodule under the coadjoint coaction (4.3).

Because $\Phi_R : \underline{H^*} \to \underline{H}$ is a morphism of braided Hopf algebras, Φ_R is a left H-module algebra and coalgebra map. Therefore, $\Phi_R(C)$ is stable under the left adjoint action of H (that is, $\Phi_R(C)$ is a normal subspace of H).

Let now $a \in H$ and write $\underline{\Delta}(a) = \underline{a_1} \otimes \underline{a_2}$. Then we have

$$\Delta(a) = \underline{a_1}R^{(2)} \otimes \mathrm{ad}_{R^{(1)}}(\underline{a_2}). \tag{4.4}$$

Indeed, writing $R = r^{(1)} \otimes r^{(2)} = R^{(1)} \otimes R^{(2)}$, relations (2.1) imply that

$$r^{(1)}R^{(1)} \otimes \mathcal{S}(R^{(2)})r^{(2)} = (\mathrm{id} \otimes \mathcal{S})(r^{(1)}R^{(1)} \otimes \mathcal{S}^{-1}(r^{(2)})R^{(2)}) = 1 \otimes 1.$$

Identity (4.4) follows after combining these with formula (4.1).

Since $C \subseteq H^*$ is a subcoalgebra, and $H^* = \underline{H^*}$ as coalgebras, then $\Phi_R(C) \subseteq \underline{H}$ is a subcoalgebra. Identity (4.4) implies that $\Delta(\Phi_R(C)) \subseteq H \otimes \Phi_R(C)$ and hence that $\Phi_R(C)$ is a normal left coideal of H, as claimed. □

Let $K_C := k[\Phi_R(C)]$ be the subalgebra of H generated by $\Phi_R(C)$. By Lemma 4.1, $K_C \subseteq H$ is a normal left coideal subalgebra of H.

Theorem 4.2. *Let $C \subseteq H^*$ be a subcoalgebra. Consider the canonical projection $\pi_C : H \to \overline{H}_C = H/HK_C^+$. Then the following hold:*

(i) *$(\overline{H}_C, \overline{R})$ is a quasitriangular quotient Hopf algebra with $\overline{R} = (\pi_C \otimes \pi_C)(R)$.*
(ii) *$K_C = H^{\mathrm{co}\,\pi_C}$ and $k[{}_R\Phi(\mathcal{S}^{-1}C)] = {}^{\mathrm{co}\,\pi_C}H$.*

(iii) $\pi_C : H \to \overline{H}_C$ *is the maximal quotient Hopf algebra with the property*

$$(p \otimes \pi_C)(Q) = p(1)1, \quad \forall p \in C. \tag{4.5}$$

Proof. By [27] HK_C^+ is a Hopf ideal and $\overline{H}_C := H/HK_C^+$ is a quotient Hopf algebra. In particular, the canonical projection gives a surjective Hopf algebra map $\pi_C : H \to \overline{H}_C = H/HK_C^+$, $h \mapsto \overline{h}$.

Clearly, the quotient $\overline{H}_C$ is quasitriangular with R-matrix $\overline{R} = (\pi_C \otimes \pi_C)(R)$. This proves (i).

The results in [27] imply in addition that $K_C = H^{\mathrm{co}\,\pi_C}$. On the other hand, ${}^{\mathrm{co}\,\pi_C}H = \mathcal{S}(H^{\mathrm{co}\,\pi_C}) = \mathcal{S}(K_C)$, cf. Section 3. In view of Lemma 2.3 and the definition of K_C, this implies that ${}^{\mathrm{co}\,\pi_C}H = k[{}_R\Phi(\mathcal{S}^{-1}C)]$. This proves (ii).

Next we prove (iii). Let $f \in \overline{H}_C^*$. Note that $f(a) = \epsilon(a)f(1)$, for all $a \in H$ such that a belongs to $H^{\mathrm{co}\,\pi_C}$ or to ${}^{\mathrm{co}\,\pi_C}H$.

Since $H^{\mathrm{co}\,\pi_C} = k[\Phi_R(C)]$ is generated by elements $p(Q^{(1)})Q^{(2)}$, $p \in C$, then

$$(p \otimes f)(Q) = p(Q^{(1)})\epsilon(Q^{(2)})f(1) = p(1)f(1).$$

Therefore π_C satisfies (4.5). To prove maximality, suppose that $t : H \to B$ is a quotient Hopf algebra such that $(p \otimes t)(Q) = 1$, for all $p \in C$. Then $t\Phi_R|_C = \epsilon 1$, implying that the left coideal subalgebra $K_C = k[\Phi_R(C)]$ is contained in $H^{\mathrm{co}\,t}$. Therefore $\ker \pi_C = HK_C^+ \subseteq H(H^{\mathrm{co}\,t})^+ = \ker t$ [27]. Thus t factorizes through π_C and maximality is established. This finishes the proof of the theorem. $\square$

Remark 4.3. Suppose C is a one dimensional subcoalgebra; that is, $C = kg$, for some $g \in G(H^*)$. Lemma 4.1 implies that $\Phi_R(C)$ is a normal left coideal of H. Since $\dim \Phi_R(C) = 1$, necessarily $\Phi_R(C) = ka$, where $a \in H$ is a central group-like element. Thus in this case we recover the result $\Phi_R(G(H^*)) \subseteq G(H) \cap Z(H)$ in [25, Theorem 2.3 (b)].

Theorem 4.2 applies in particular when $C \subseteq H^*$ is a Hopf subalgebra. The case $C = H^*$ is contained in the following theorem. It is of special interest, as it is related with the notion of *modularization* studied in [4, 18]; c.f. Subsection 4.2.

Theorem 4.4. *Let $K = \Phi_R(H^*) \subseteq H$. Then K is a normal left coideal subalgebra of H.*

Consider the canonical projection $\pi : H \to \overline{H} = H/HK^+$. Then the following hold:

(i) *$(\overline{H}, \overline{R})$ is a triangular quotient Hopf algebra with R-matrix $\overline{R} = (\pi \otimes \pi)(R)$.*

(ii) *$\Phi_R(H^*) = H^{\operatorname{co}\pi}$ and ${}_R\Phi(H^*) = {}^{\operatorname{co}\pi}H$.*
(iii) *$\pi : H \to \overline{H}$ is the maximal quotient Hopf algebra with the property*

$$(\pi \otimes \operatorname{id})(Q) = 1 = (\operatorname{id} \otimes \pi)(Q). \tag{4.6}$$

Note that finite dimensional triangular Hopf algebras have been completely classified [10, 2, 12]. It turns out that $(\overline{H}, \overline{R})$ is necessarily a twisting of a modified supergroup algebra. We shall use this fact in later applications of this theorem. See Section 5.

Proof. Parts (ii) and (iii) are special instances of Theorem 4.2. Indeed, $\Phi_R(H^*)$ is a subalgebra of H. To prove part (i) it is enough to show that $\overline{H}$ is triangular. We have $\Phi_{\overline{R}} = \pi\Phi_R\pi^*$, where $\pi^* : \overline{H}^* \hookrightarrow H^*$ is the dual inclusion. Hence $\Phi_{\overline{R}} = \epsilon 1$, because $\pi|_K = \epsilon 1$. Therefore $(\overline{H}, \overline{R})$ is triangular, as claimed. □

Remark 4.5. Let $C \subseteq H^*$ be a subcoalgebra. Keep the notation in Theorems 4.2, 4.4.

(1) Since $\Phi_R(H^*) \subseteq H_R$ [22], then the left coideal $\Phi_R(C)$ and also $K_C = k[\Phi_R(C)]$ are contained in H_R. Moreover, since $Q \in H_R \otimes H_R$, then $\Phi_R(H^*) = \Phi_R(H_R^*)$. In particular, whenever H_R is factorizable, $\Phi_R(H^*) = H_R$ is a normal Hopf subalgebra of H.

(2) The inclusion $K_C \subseteq H_R$ also implies that $[H : H_R]$ divides $\dim \overline{H}_C$.

Indeed, consider the exact sequences $K_C \subseteq H \to \overline{H}_C$, $K_C \subseteq H_R \to (\overline{H_R})_C$. Taking dimensions we get

$$\begin{aligned} \dim H = \dim H_R[H : H_R] &= \dim K_C \dim(\overline{H_R})_C\, [H : H_R] \\ &= \dim K_C \dim \overline{H}_C. \end{aligned}$$

Hence $\dim \overline{H}_C = \dim(\overline{H_R})_C\, [H : H_R]$, and $[H : H_R]/ \dim \overline{H}_C$, as claimed.

(3) The maximality of the quotient $H \to \overline{H}_C$ implies that there is a sequence of surjective Hopf algebra maps $H \to \overline{H}_C \to \overline{H}$.

Lemma 4.6. *Let $C \subseteq H^*$ be a subcoalgebra.*

(i) *$\overline{H}_C = H$ if and only if $\Phi_R|_C = \epsilon$ if and only if $C \subseteq \overline{H}^*$.*
(ii) *If $\overline{H}_C = k$ then H is factorizable.*
(iii) *$k[C] \supseteq \overline{H}_C^*$ if and only if $\overline{H}_C$ is triangular.*

In particular, if H is not factorizable and $C \nsubseteq \overline{H}^*$, then $\overline{H}_C$ is a proper quotient Hopf algebra.

Proof. Part (i) follows from the definition of $\overline{H}_C$. Part (ii) follows from Remark 4.5 (3). Finally, part (iii) is a consequence of Theorem 4.2 (iii). $\square$

Theorem 4.7. *Suppose $A \subseteq H^*$ is a Hopf subalgebra. Then there is a Hopf subalgebra $B \subseteq H^*$ such that*

(i) $[H^* : A]$ *divides* $\dim B$.
(ii) $\Phi_R|_{A \cap B} = \epsilon$. *Furthermore, $B = H^*$ if and only if $\Phi_R|_A = \epsilon$.*

Proof. Apply Theorem 4.2 to the subcoalgebra A. Then $\Phi_R(A)$ is a subalgebra of H and thus $K_A = \Phi_R(A)$.

Moreover, $\Phi_R : A \to \Phi_R(A)$ is a surjective map of braided Hopf algebras over H. Hence $\dim \Phi_R(A)/\dim A$ by the Nichols–Zoeller theorem applied to the corresponding biproducts.

Consider the quotient $\overline{H}_C = H/HK_C^+$. So that $\dim \overline{H}_C = \frac{\dim H}{\dim \Phi_R(A)}$. Let $B = \overline{H}^*_A \subseteq H^*$. Then B satisfies (i) and (ii). $\square$

The following application of Theorem 4.2 gives restrictions on the possible quotients of a factorizable Hopf algebra.

Theorem 4.8. *Let (H, R) be a factorizable Hopf algebra. Suppose $A \subseteq H^*$ is a Hopf subalgebra. Then there is a Hopf subalgebra $B \subseteq H^*$ such that*

(i) $\dim A \dim B = \dim H$.
(ii) $A \cap B = k1$.

Proof. In this case $\dim \Phi_R(A) = \dim A$ because H is factorizable. As in the proof of Theorem 4.7, take for B the Hopf subalgebra $\overline{H}^*_A \subseteq H^*$. Then B satisfies (i). Moreover, we have $\Phi_R|_{A\cap B} = \epsilon$, whence $A \cap B = k1$, since H is factorizable by assumption. Thus B satisfies also (ii). $\square$

Part (ii) of the following proposition generalizes [25, Theorem 2.3 (b)]. Part (iii) gives a refinement of [22, Proposition 3 c)].

Proposition 4.9. *Identify $\overline{H}^*$ with a Hopf subalgebra of H^*. Then*

(i) Φ_R *induces an isomorphism* $\Phi_R : H^*/(\overline{H}^*)^+H^* \xrightarrow{\simeq} K$.
(ii) Φ_R *induces an injective group homomorphism*

$$\Phi_R : G(H^*)/G(\overline{H}^*) \hookrightarrow G(H) \cap Z(H).$$

(iii) *Let $\alpha \in G(H^*)$ be the modular element. Then $\alpha \in G(\overline{H}^*)$.*

In particular, if $G(\overline{H}^*) = 1$, then H is unimodular.

Proof. (i). We shall see that $\ker \Phi_R = (\overline{H}^*)^+ H^*$, whence the claimed isomorphism. We know that

$$\dim K = \frac{\dim H}{\dim \overline{H}} = \dim(H^*/(\overline{H}^*)^+ H^*) = \dim H^* - \dim(\overline{H}^*)^+ H^*.$$

Thus it will be enough to see that $(\overline{H}^*)^+ H^* \subseteq \ker \Phi_R$. For this, let $p \in (\overline{H}^*)^+$, $f \in H^*$. So that $\Phi_R(p) = \epsilon(p) = 0$. Using (QT1), (QT3), we have

$$\begin{aligned} \phi_R(pf) &= (pf)(r^{(2)}R^{(1)})r^{(1)}R^{(2)} \\ &= (p \otimes f)(\Delta(r^{(2)})\Delta(R^{(1)}))r^{(1)}R^{(2)} \\ &= (p \otimes f)(s^{(2)}R^{(1)} \otimes r^{(2)}t^{(1)})r^{(1)}s^{(1)}R^{(2)}t^{(2)} \\ &= f(r^{(2)}t^{(1)})r^{(1)}\Phi_R(p)t^{(2)} = 0, \end{aligned}$$

where $R = r = s = t$. This gives the desired inclusion.

(ii). The irreducible character χ of H comes from an irreducible character of $\overline{H}$ if and only if $\Phi_R(\chi) = \deg \chi$. Thus $\ker \Phi_R|_{G(H^*)} = G(\overline{H}^*)$.

(iii). By [22] we have $\Phi_R(\alpha) = 1$. Using (ii), we get $\alpha \in G(\overline{H}^*)$. □

We next give some sufficient conditions for the map π to be normal.

Proposition 4.10. *Suppose that either one of the following conditions hold:*

(a) $H_+ \subseteq K$,
(b) $R_{21}R = RR_{21}$,
(c) $\overline{R} = 1 \otimes 1$ *and* K *is commutative, or*
(d) $\overline{R} = 1 \otimes 1$ *and* $\dim K$ *is relatively prime to* $\dim \overline{H}$.

Then $K \subseteq H$ *is a normal Hopf subalgebra and there is an exact sequence of Hopf algebras*

$$k \to K \hookrightarrow H \xrightarrow{\pi} \overline{H} \to k.$$

Proof. Suppose (a) holds. Then the conclusion follows from Corollary 3.6.

Now assume (b). Then $H^{\operatorname{co}\pi} = {}^{\operatorname{co}\pi}H$, in view of Theorem 4.4 (ii). Hence π is normal as claimed.

If (c) holds, then $f_R(\overline{H}^*) \subseteq K = H^{\operatorname{co}\pi}$, and the conclusion follows from Proposition 3.10.

Finally, in case (d), necessarily $f_R|_{\overline{H}^*} = \epsilon$, and the conclusion follows from Remark 3.11. □

Theorem 4.11. *Suppose that $R \in kG(H) \otimes kG(H)$. Then $\Phi_R(H^*) \subseteq H$ is a commutative normal Hopf subalgebra, and there is an exact sequence of Hopf algebras*

$$k \to \Phi_R(H^*) \to H \xrightarrow{\pi} \overline{H} \to k.$$

In particular, H is an extension of a dual group algebra by a twisting of a modified supergroup algebra.

Proof. It will be enough to show that $\Phi_R(H^*)$ is a Hopf subalgebra. Commutativity of $\Phi_R(H^*) \subseteq (kG(H))_R$ will follow from Example 2.1. The assumption $R \in kG(H) \otimes kG(H)$ is equivalent to $H_R \subseteq kG(H)$. If this holds, then $\Phi_R(H^*) \subseteq H_R \subseteq kG(H)$, and therefore $\Phi_R(H^*)$ is necessarily a Hopf subalgebra, as claimed. □

Example 4.12. Quasitriangular structures on nonsemisimple Hopf algebras of dimension 8 have been classified in [28]. It turns out that there is only one example, denoted A_{C_2}, of such a Hopf algebra which admits an R-matrix that is not triangular.

Explicitly, $H = A_{C_2}$ is presented by generators g, x, y and relations $g^2 = 1$, $x^2 = y^2 = 0$, $gx = -xg$, $gy = -yg$, $xy = -yx$, with $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes g + 1 \otimes x$, $\Delta(y) = y \otimes g + 1 \otimes y$. The R-matrices on H are parameterized by R_{abcd}, with $a, b, c, d \in k$, such that R_{abcd} is minimal whenever $b \neq 0$ or $c \neq 0$ or $ad \neq 0$.

By [28, Proof of Proposition 2.9], for $R = R_{abcd}$, we have

$$R_{21}R = 1 \otimes 1 + (b-c)(y \otimes x - x \otimes y)(1 \otimes g) - (b-c)^2 xy \otimes xy.$$

Therefore, H is triangular if $b = c$, and otherwise $\Phi_R(H^*)$ is a left coideal subalgebra of dimension 4 spanned by $1, xg, yg, xy$. In particular, $\Phi_R(H^*)$ is not a Hopf subalgebra of H.

Example 4.13. Let $p \neq q$ be prime numbers such that $p = 1 \pmod q$. Let A_i, $0 \leq i \leq p-1$, be one of the nontrivial self-dual semisimple Hopf algebras with $\dim A_i = pq^2$ studied in [13, 3]. Quasitriangular structures in A_i's were discussed in [13]. We list some properties of these Hopf algebras (see *loc. cit.* for a proof):

(i) $G(A_0) \simeq \mathbb{Z}_q \times \mathbb{Z}_q$ and $G(A_i) \simeq \mathbb{Z}_{q^2}$, $i > 0$.

(ii) $G(A_i) \cap Z(A_i) \simeq \mathbb{Z}_q$, and this is the only normal Hopf algebra quotient of dimension q of A_i^*.

(iii) Suppose $i > 0$. Then A_i is quasitriangular if and only if $q = 2$. Moreover, when $q = 2$, A admits $2p-2$ minimal quasitriangular structures, none of them triangular.

(iv) A_0 admits triangular structure with $(A_0)_R = kG(A_0)$. A_0 is minimal quasitriangular if and only if $q = 2$ and in this case none of the quasitriangular structures is triangular.

(v) The proper Hopf subalgebras of A_i are commutative or cocommutative. In particular, if A_i admits a quasitriangular structure which is not minimal, then $(A_i)_R \subseteq kG(A_i)$.

Furthermore, A_i has a unique normal quotient of dimension q; thus a unique Hopf subalgebra of dimension pq (which is necessarily normal, because q is the smallest prime number dividing $\dim A$). Hence if (A_i, R) is minimal quasitriangular, necessarily $(A_i)_+ = (A_i)_- = A$. That is, $f_R : A_i^{*\,\mathrm{cop}} \to A_i$ is an isomorphism.

We prove next the following additional properties.

Lemma 4.14. *Let $A = A_i$, $0 \leq i \leq p-1$, and assume (A, R) is quasitriangular. Then we have*

(vi) *A is not factorizable.*

Suppose q is odd and (A, R) is not triangular. Then

(vii) *$\Phi_R(A^*)$ is a commutative (normal) Hopf subalgebra of dimension q or pq.*

(viii) *$G(A) \cap Z(A) \subseteq \Phi_R(A^*)$.*

Proof. (vi). Suppose on the contrary that A is factorizable. By Proposition 4.9, $|G(\overline{A}^*)| = q$. This implies that $\overline{A} \neq 1$ and hence that A is not factorizable, thus proving (vi).

Now suppose q is odd and (A, R) is not triangular. Let $K = \Phi_R(A^*)$ and consider the sequence $K \subseteq A \to \overline{A}$. By (vi), $\overline{A} \neq 1$, therefore $\dim \overline{A} = q^2, pq$ or q. By (iii) and (iv) A is not minimal. Then by (v) and Theorem 4.11 K is a (commutative) normal Hopf subalgebra of A. Necessarily $\dim K = q$ or pq, thus part (vii) follows.

We have $\overline{H} \simeq kF$ for some group F with $|F| = q$ or pq. In particular, $G(k^F)$, being a subgroup of $G(A^*) \simeq \mathbb{Z}_q \times \mathbb{Z}_q$, is of order q.

By Proposition 4.9 (i), Φ_R induces an injective group homomorphism $G(A^*)/G(k^F) \to G(A) \cap Z(A)$. Since $G(A^*)/G(k^F)$ is not trivial and $G(A) \cap Z(A)$ is of order q, this homomorphism is in fact an isomorphism $G(A^*)/G(k^F) \simeq G(A) \cap Z(A)$. This proves (viii). □

4.1. **Minimality.** The following lemma gives a relation between the minimal quasitriangular Hopf subalgebras of H and $\overline{H}$.

Lemma 4.15. *If $\overline{H}$ is minimal, then H is minimal.*

Proof. Consider the sequence $H_R \subseteq H \xrightarrow{\pi} \overline{H}$. Since $\Phi_R(H^*) \subseteq H_R$,

$$H_R^{\mathrm{co}\,\pi} = H_R \cap H^{\mathrm{co}\,\pi} = H_R \cap \Phi_R(H^*) = \Phi_R(H^*) = \Phi_R(H_R^*).$$

Therefore $\pi(H_R) \simeq H_R/\Phi_R(H_R^*)^+H_R = \overline{H_R}$. Hence we have exact sequences

$$\Phi_R(H^*) \subseteq H \to \overline{H}, \quad \Phi_R(H^*) \subseteq H_R \to \overline{H_R}.$$

Taking dimensions we get

$$\dim H = \dim \Phi_R(H^*) \dim \overline{H}, \quad \dim H_R = \dim \Phi_R(H^*) \dim \pi(H_R).$$

Since $\overline{R} \in \pi(H_R) \otimes \pi(H_R)$, then $\overline{H}_{\overline{R}} \subseteq \pi(H_R)$. Now assume that $\overline{H}$ is minimal; then $\overline{H} = \overline{H}_{\overline{R}} = \pi(H_R)$. Therefore $\dim H = \dim H_R$ and $H = H_R$ is minimal, as claimed. □

4.2. **Modularization ([4, 18].)** The quotient $\pi : H \to \overline{H}$ is an analogue of the *transparent* tensor subcategory $\mathcal{T}_{\mathcal{C}}$ of the category $\mathcal{C} = \operatorname{Rep} H$ [4, pp. 224].

Indeed, if H is semisimple, then this follows from relations in Theorem 4.4 (iii). However, our construction does not assume semisimplicity nor even existence of a ribbon structure.

Assume that H is semisimple. Then the category H-mod is a *premodular category*; here the braiding is given by the action of the R-matrix:

$$\sigma_{V,W}(v \otimes w) = R^{(2)}w \otimes R^{(1)}v, \quad v \in V,\, w \in W.$$

The ribbon structure is given by the action of the (central) Drinfeld element $u = \mathcal{S}(R^{(2)})R^{(1)}$.

Recall from [4] that a left H-module V is called *transparent* if for any H-module W, $\sigma_{W,V}\sigma_{V,W} = \mathrm{id}$. Observe that (H, R) is triangular if and only if all simple H-modules are transparent, and (H, R) is factorizable if and only if H has no non-trivial irreducible transparent simple modules.

Let $R(H) \subseteq H^*$ be the character algebra of H. Condition (QT5) implies that $R(H)$ is a commutative semisimple algebra over k. Let $\mathcal{T}(H) \subset R(H)$ denote the linear span of the characters of the irreducible transparent H-modules.

Proposition 4.16. *$\mathcal{T}(H) = R(\overline{H}) \subseteq R(H)$.*

Proof. The irreducible character $\chi \in R(H)$ is transparent if and only if $\Phi_R(\chi) = \chi(1)1$. The proof follows from Theorem 4.4 (iii). □

By [9], after modifying if necessary the R-matrix $\overline{R}$, $\overline{H}$ is isomorphic to a twisting of a group algebra $H \simeq (kG)_J$, where G is a finite group and $J \in kG \otimes kG$ is a normalized 2-cocycle.

Remark 4.17. Note that $\operatorname{Rep} H$, with its canonical braiding, is modularizable in the sense of [4] if and only $\overline{u} = 1$, where $\overline{u}$ is the Drinfeld element of $\overline{H}$, that coincides with the image of the Drinfeld element $u \in H$ under the canonical projection $H \to \overline{H}$. Indeed the last condition amounts to the category of transparent objects in $\operatorname{Rep} H$ being tannakian [6, 9].

5. Application

In this section we apply the results in Sections 3 and 4 to the classification of finite dimensional Hopf algebras with square-free dimension.

In what follows we shall consider a quasitriangular Hopf algebra (H, R). Assume that $\dim H$ is odd and square-free.

Lemma 5.1. *Suppose (H, R) is triangular. Then H is semisimple and $R = 1 \otimes 1$. In particular, H is isomorphic to a group algebra.*

Proof. It follows from [12, Theorem 4.3] that H has the Chevalley property; in other words, the coradical of the dual Hopf algebra H^* is a Hopf subalgebra. By [2], H is twist equivalent to a triangular Hopf algebra H' with R-matrix R' of rank ≤ 2.

Let $u' \in {H'}_{R'}$ be the Drinfeld element. Since H' is triangular, $u' \in G(H')$. Moreover, ${u'}^2 = 1$, since $\operatorname{rk} R' \leq 2$. Because $\dim H' = \dim H$ is odd, we get $u' = 1$. Hence $\mathcal{S}^2 = \operatorname{ad}_{u'} = \operatorname{id}$, implying that H', and therefore also H, are semisimple.

Because H is odd dimensional, its Drinfeld element u is necessarily trivial (otherwise it would be a group-like element of order 2, by [9, Lemma 2.1]). By [11, Corollary 2.2.2], $R = 1 \otimes 1$, since $\dim H_R$ must be a square dividing $\dim H$. □

Lemma 5.2. *We have $\Phi_R(H^*) = H_R$ is a normal Hopf subalgebra of H and there is an exact sequence of Hopf algebras*

$$k \to H_R \to H \xrightarrow{\pi} \overline{H} \to k.$$

Proof. Consider the projection $\pi : H \to \overline{H}$ and let $K = H^{\operatorname{co} \pi} = \Phi_R(H^*)$. By Lemma 5.1, $\overline{R} = 1 \otimes 1$ and therefore $f_R(\overline{H}^*) \subseteq K$.

Because $\dim H$ is square-free, $\dim \overline{H}$ and $\dim H^{\operatorname{co} \pi}$ are relatively prime. This implies that $f_R(\overline{H}^*) = k1$ since, by the Nichols-Zoeller Theorem, $\dim f_R(\overline{H}^*)$ divides both $\dim \overline{H}$ and $\dim H^{\operatorname{co} \pi}$. In view of Proposition 3.10, π is normal (c. f. Remark 3.11).

It remains to show that $K = H_R$ or, in other words, that H_R is factorizable. This is in turn equivalent to proving that $\overline{H_R} = k$. Since $\dim H_R$ is square-free and $\dim H_R / (\dim H_+)^2$, we have $H_+ = H_- = H_R$. In particular, R induces an isomorphism $f_R : H_R^{* \operatorname{cop}} \to H_R$.

Since $\overline{R} = 1 \otimes 1$, as in the previous paragraph, we get $f_R(\overline{H_R}^*) = k1$. This implies that $\overline{H_R} = k$ as claimed. The proof of the lemma is now complete. □

Proposition 5.3. *H is semisimple.*

Proof. If (H, R) is triangular, the proposition follows from Lemma 5.1. Suppose next that (H, R) is factorizable. Then H is minimal quasitriangular and therefore, $\dim H$ being square-free, $\dim H = \operatorname{rk} R$. Thus H is semisimple, by Lemma 2.6. The general case follows from Lemma 5.2, since an extension of semisimple Hopf algebras is semisimple. □

Proof of Theorem 1.2. By [25, Theorem 3.2] the square of the dimensions of the irreducible H_R-modules divide $\dim H_R$. Since $\dim H_R$ is square-free all irreducible H_R-modules must be one-dimensional and therefore H_R is commutative.

Hence $H_R \simeq k^\Gamma$ for some finite group Γ. In addition Γ must be abelian (hence cyclic) because k^Γ is quasitriangular.

In view of Lemma 5.2 we have an *abelian* extension [15]

$$k \to k^\Gamma \to H \to kF \to k,$$

where Γ is cyclic and $R \in k^\Gamma \otimes k^\Gamma$. This extension gives rise to compatible actions $\triangleright : \Gamma \times F \to F$ and $\triangleleft : \Gamma \times F \to \Gamma$ and 2-cocycles $\sigma : F \times F \to (k^\Gamma)^\times$, $\tau : \Gamma \times \Gamma \to (k^F)^\times$ in such a way that H is a bicrossed product $H \simeq k^{\Gamma\,\tau}\#_\sigma kF$. Moreover, since Γ is cyclic we may assume (up to Hopf algebra isomorphisms) that $\tau = 1$.

In the basis $(e_s x : s \in \Gamma,\, x \in F)$, the comultiplication and multiplication of $k^{\Gamma\,\tau}\#_\sigma kF$ are determined by

$$\Delta(e_g x) = \sum_{st=g} e_s\#(t \triangleright x) \otimes e_t\#x, \tag{5.1}$$

$$(e_s\#x)(e_t\#y) = \delta_{s\triangleleft x,t}\, e_s\sigma(x,y)\#xy.$$

On the other hand the R-matrix $R \in k^\Gamma \otimes k^\Gamma$ can be written in the form $R = \sum_{s,t}\langle s,t\rangle e_s \otimes e_t$, for some (non-degenerate) bicharacter $\langle\,,\,\rangle : \Gamma \times \Gamma \to k^\times$.

Let $x \in F$. We have $\Delta^{\mathrm{cop}}(1\#x) = \sum_{s,t} e_t\#x \otimes e_s\#(t \triangleright x)$.

On the other hand,

$$R\Delta(1\#x)R^{-1} = \sum_{s,t}\langle s,t\rangle\langle s \triangleleft (t \triangleright x), t \triangleleft x\rangle^{-1} e_s\#(t \triangleright x) \otimes e_t\#x.$$

Comparing both expressions we find that the action $\triangleright : \Gamma \times F \to F$ must be trivial. This implies that H is cocommutative in view of (5.1).

So $H \simeq kG$, and G fits into an extension $1 \to \widehat{\Gamma} \to G \to F \to 1$. Then $G \simeq \widehat{\Gamma} \rtimes F$ by the Schur-Zassenhaus' Lemma.

Also $(g^{-1} \otimes g^{-1})R(g \otimes g) = R$, for all $g \in G$, implying that the form $\langle \, , \, \rangle$ is F-invariant. This finishes the proof of the theorem. $\square$

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